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ATTRACTIVENESS OF INTERACTIONS FOR BINARY LATTICE SYSTEMS AND THE GLOBAL MARKOV PROPERTY

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We impose a necessary and sufficient condition for the existence of an order on the states of a binary lattice system that makes a given interaction attractive. This condition is automatically satisfied in the case of a translation invariant interaction on a square lattice. So, in this case, the global Markov property holds under uniqueness.

attractive interactions * binary lattice systems * global Markov property

Introduction

The global Markov property for binary lattice systems has been demonstrated under certain conditions on the interaction. One of these conditions ("strong uniqueness" [1, 2]) asks for the interaction to be weak enough that there exists only one Gibbs measure, even if we impose conditions on the spins of an arbitrary subset of sites. It would be interesting to find out when the global Markov property holds if we only assume uniqueness of the "unconditioned" Gibbs measures. This is strictly weaker than strong uniqueness. Föllmer [3] points out that the global Markov property is true in the case of uniqueness, if we have an attractive interaction. This result applies also to the "repulsive" case by reversing the underlying order structure on a "chessboard" set of sites. The open problem of the global Markov property under uniqueness is also raised in Israel [4]. This paper contains an example of a translation invariant interaction, which admits two different extremal Gibbs measures both violating the global Markov property. Nonuniqueness is essential in these examples.

Our aim here is to investigate when a given nearest neighbour interaction admits an order structure that makes it attractive. We establish a necessary and sufficient

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condition in terms of graph theoretic notions. Roughly, this condition demands that the pattern of bonds along which the interaction is repulsive should be "regular".

This condition is always satisfied if we have a translation invariant nearest neighbour interaction on a square lattice. So, in this case we may conclude the global Markov property for the Gibbs state in the case of uniqueness.

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Preliminaries

We briefly review the basic definitions, and describe the notation we shall use. For more background on Gibbs measures and lattice systems, see e.g. [5].

Let Γ be a *d-dimensional lattice*, i.e., a graph that is embeddable in $\mathbb{R}^d$, and which is "well behaved", i.e., each vertex is linked only with finitely many others; the length of the edges is bounded, and each finite volume contains only finitely many vertices. Let $\mathcal{V}$ denote the set of all edges (or bonds) V in Γ. Two vertices (or sites) γ, γ' are called *nearest neighbours* if $(\gamma, \gamma') \in \mathcal{V}$.

Put $S = \{0, 1\}$, $\Omega = S^\Gamma$, $\mathcal{B}$ = product-σ-algebra on Ω. For $W \subseteq \Gamma$ put $\Omega_W = S^W$, $\mathcal{B}_W = \{A \times \Omega_{\Gamma\setminus W} \mid A \in \mathcal{B}(\Omega_W)\}$. We frequently identify $\mathcal{B}_W$ with $\mathcal{B}(\Omega_W)$.

A *nearest neighbour interaction* Φ is a system $\Phi = (\Phi_V)_{V\in\mathcal{V}}$ where $\Phi_V : \Omega_V \to \mathbb{R} \cup \{\infty\}$. For technical reasons we need that for all $V = \{\gamma, \gamma'\}$ and all $\omega \in S$, there exists $\nu \in S$ such that $\Phi_V(\omega, \nu) \neq \infty$. For $\Lambda \subseteq \Gamma$ define the *boundary* $\partial\Lambda$ of Λ, by $\partial\Lambda = \{\gamma \notin \Lambda \mid \exists \gamma' \in \Lambda : (\gamma, \gamma') \in \mathcal{V}\}$.

For finite Λ and configurations $\tau \in \Omega_{\Lambda^c}$ let $\rho_{\Lambda,\tau}$ be the measure on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$ with density $\omega \mapsto \exp(-\sum \Phi_V(\omega\hat{}\tau) \mid V \cap \Lambda \neq \emptyset, V \in \mathcal{V}\})$ (up to normalization) w.r.t. uniform counting measure, where $\hat{}$ denotes concatenation. $\rho_{\Lambda,\tau}$ only depends on $\tau \restriction \partial\Lambda$. Call a probability measure P on $(\Omega, \mathcal{B})$ a *Gibbs measure* if it satisfies the following equilibrium condition: $P(B \mid \tau \text{ on } \Lambda^c) = \rho_{\Lambda,\tau}(B)$ for all finite Λ and $B \in \mathcal{B}_\Lambda$. Let $\mathcal{G}$ denote the set of all Gibbs measures (w.r.t. the given interaction Φ).

$P \in \mathcal{G}$ has the *global Markov property* if for all $\Lambda \subseteq \Gamma$ (not necessarily finite), $\mathcal{B}_\Lambda$ and $\mathcal{B}_{\Gamma\setminus(\Lambda\cup\partial\Lambda)}$ are conditionally independent under P, given $\mathcal{B}_{\partial\Lambda}$.

Now suppose that an order $\leqslant_\gamma$ on S is given for each $\gamma \in \Gamma$, and let $\leqslant$ be the corresponding (partial) product order $\prod_{\gamma\in\Gamma} \leqslant_\gamma$. Let $\mathcal{M}^\uparrow_\leqslant = \{f \geqslant 0 \mid f$ depends only on finitely many coordinates and is monotone w.r.t. $\leqslant\}$. Then Φ is called $\leqslant$-attractive if for all finite Λ and all $f \in \mathcal{M}^\uparrow_\leqslant$, the function $\tau \mapsto E_{\rho_{\Lambda,\tau}}(f)$ is monotonically increasing. The order $\leqslant$ induces the following dual order $\circledleq$ on $\mathcal{G}$: $P \circledleq Q$ iff $E_P(f) \leqslant E_Q(f)$ for all $f \in \mathcal{M}^\uparrow_\leqslant$.

Fact 1 (Föllmer [3]). *Let Φ be a $\leqslant$-attractive nearest neighbour interaction. Then there exist unique $\leqslant$-maximal and -minimal Gibbs states; both have the global Markov property.*

Fact 2 (Preston [5, Chapter 9]). *Let Φ and $\leqslant$ have the following property: For all $V \in \mathcal{V}$ and $\omega, \omega' \in \Omega_V$:*

$$\Phi_V(\omega \wedge \omega') + \Phi_V(\omega \vee \omega') \leqslant \Phi_V(\omega) + \Phi_V(\omega'), \tag{$*$}$$

where $\wedge$ and $\vee$ are w.r.t. $\leqslant$. Then Φ is $\leqslant$-attractive.

Preston's result, using natural order, immediately generalizes to arbitrary product orders. Fact 2 holds in a much more general setting: Φ does not need to be nearest neighbour, and S not be $\{0, 1\}$. But under our restrictions everything becomes very easy: $(*)$ needs only to be checked when for $V = (\gamma, \gamma')$, ω and ω' are incomparable under $\leqslant_\gamma \times \leqslant_{\gamma'}$. In the case of equal order structures on γ, γ' this yields $\{(\omega \wedge \omega'), (\omega \vee \omega')\} = \{(00), (11)\}$; $\{\omega, \omega'\} = \{(01), (10)\}$, and $(*)$ becomes

$$\Phi_V(00) + \Phi_V(11) \leqslant \Phi_V(01) + \Phi_V(10). \tag{$*, =$}$$

In the case of nonequal order structures on γ, γ' we similarly get

$$\Phi_V(00) + \Phi_V(11) \geqslant \Phi_V(01) + \Phi_V(10). \tag{$*, \neq$}$$

But since these two inequalities cover all possible cases, we "locally" have always a suitable order that ensures attractiveness. If $\# S > 2$ then this is no longer true.

Notice that the converse to fact 2 also holds in the case $\# S = 2$:

Lemma 3. *Given a $\leqslant$-attractive interaction Φ, then Φ satisfies $(*)$.*

Proof. W.l.o.g. let $\leqslant$ be the natural order. Assume the contrary to $(*)$. Then there exists $V = \{\gamma, \gamma_1\} \in \mathcal{V}$ such that $b_{00}b_{11} < b_{01}b_{10}$, where for $\omega \in \Omega_V$, $b_\omega = \exp(-\Phi_V(\omega)) \geqslant 0$. Let $\gamma_2, \ldots, \gamma_n$ be the remaining neighbours of γ. Put $f(\omega) = \omega_\gamma$. So, $f \in \mathcal{M}^{\uparrow}_{\leqslant}$. For $j = 2, \ldots, n$, choose $\sigma_j \in \{0, 1\}$ such that $\Phi_{(\gamma, \gamma_j)}(0, \sigma_j) < \infty$. Put, for $i = 0$, 1 and $j = 2, \ldots, n$, $(\tau_i)_{\gamma_j} = \sigma_j$, and $(\tau_i)_{\gamma_1} = i$. So, $\tau_0 \leqslant \tau_1$. Define, for $i = 0, 1$, $z_i = \exp(-\sum \Phi_{(\gamma, \gamma_j)}(i, \sigma_j) \mid j = 2, \ldots, n\})$. By the choice of the σ_j, $z_1 \geqslant 0$ and $z_0 > 0$. Then

$$\begin{aligned} E_{\rho_{\{\gamma\},\tau_1}}(f) / E_{\rho_{\{\gamma\},\tau_0}}(f) &= [b_{11}(b_{10}z_1 + b_{00}z_0)]/[b_{10}(b_{11}z_1 + b_{10}z_0)] \\ &= [z_0 b_{11} b_{00} + z_1 b_{11} b_{10}]/[z_0 b_{01} b_{10} + z_1 b_{10} b_{11}] \\ &= 1 - [z_0(b_{01}b_{10} - b_{11}b_{00})]/[z_0 b_{01} b_{10} + z_1 b_{10} b_{11}] \\ &< 1 \end{aligned}$$

by our assumption on the b_ω. Φ is not $\leqslant$-attractive.

In order to formulate our result we have to adopt some definitions from graph theory.

Definition 4. Let $(\Gamma, \mathcal{V})$ be a graph and $\mathcal{T} \subseteq \mathcal{V}$ be a subset of edges. Denote by $(\Gamma/\mathcal{T}, \mathcal{V}/\mathcal{T})$ the quotient graph in which two vertices γ, γ_1 are identified if there is a finite path of edges from $\mathcal{T}$ that links γ and γ_1. Let $[\gamma]_\mathcal{T}$ denote the vertex of $\Gamma/\mathcal{T}$ corresponding to γ. In general, $\Gamma/\mathcal{T}$ is no longer locally finite.

A graph $(\Gamma, \mathcal{V})$ is n-colourable if there exists a function $c : \Gamma \to \{1, \dots, n\}$ such that for each $V = (\gamma, \gamma') \in \mathcal{V}$, $c(\gamma) \neq c(\gamma')$.

We need another definition which deals with the "local attractivity structure" of a given interaction Φ.

Definition 5. An edge $V \in \mathcal{V}$ is called *attractive* if $(*, =)$ holds but not $(*, \neq)$; it is called *repulsive* if $(*, \neq)$ holds but not $(*, =)$. In the remaining case call V *indifferent*. Let the corresponding subsets of $\mathcal{V}$ be denoted by $\mathcal{A}$, $\mathcal{R}$, and $\mathcal{I}$, respectively.

Theorem 6. *Let an interaction Φ be given. Then the following are equivalent:*

(a) *There exists a partial order $\leq = \prod_{\gamma \in \Gamma} \leq_\gamma$ that makes Φ attractive.*

(b) *There exists $\mathcal{T} \subseteq \mathcal{V}$ such that $\mathcal{A} \subseteq \mathcal{T}$, $\mathcal{R} \subseteq \mathcal{T}^c$, and $\Gamma / \mathcal{T}$ is 2-colourable.*

Proof. (b)$\Rightarrow$(a): Let $\Gamma/\mathcal{T}$ be coloured by the colours pink and grey. Then define $\leq_\gamma$ as the natural (respectively reverse) order on $\{0, 1\}$ if the colour of $[\gamma]_{\mathcal{T}}$ is pink (respectively grey). We have to show $(*)$ for all $V \in \mathcal{A} \cup \mathcal{R}$ (in the case of $V \in \mathcal{I}$, $(*)$ is clearly satisfied).

Let $V = (\gamma, \gamma') \in \mathcal{A}$. Then $[\gamma]_{\mathcal{T}} = [\gamma']_{\mathcal{T}}$ and hence γ and γ' carry the same order. Since $V \in \mathcal{V}$, $(*)$ holds.

Similarly, let $V = (\gamma, \gamma') \in \mathcal{R}$. Then $[\gamma]_{\mathcal{T}} \neq [\gamma']_{\mathcal{T}}$, but both are nearest neighbours in $\Gamma/\mathcal{T}$, so they carry different colours. But this means that γ and γ' carry different orders, i.e., $(*)$ holds since $V \in \mathcal{V}$.

(a)$\Rightarrow$(b): Let a partial order $\leq$ on Γ be given that makes Φ $\leq$-attractive. Define $\mathcal{T} = \{V \in \mathcal{V} \mid \text{both sites in } V \text{ carry the same order}\}$. We have to show: (1) $\mathcal{A} \subseteq \mathcal{T}$, $\mathcal{R} \subseteq \mathcal{T}^c$, and (2) $\Gamma/\mathcal{T}$ is 2-colourable.

(1) Let $V = (\gamma, \gamma')$ be given. By Lemma 3, $(*)$ holds. If $V \in \mathcal{A}$, $(*, =)$ holds but not $(*, \neq)$. This ensures that γ and γ' carry the same order, so $V \in \mathcal{T}$. If $V \in \mathcal{R}$ then similarly $V \in \mathcal{T}^c$.

(2) Define the colour of $[\gamma]_{\mathcal{T}}$ to be pink (respectively grey) if the order at γ is natural (respectively reverse). By definition of $\mathcal{T}$, this is independent of the choice of the representative. Now, let $([\gamma]_{\mathcal{T}}, [\gamma']_{\mathcal{T}}) \in \mathcal{V}/\mathcal{T}$. Then there are $\gamma_1 \in [\gamma]_{\mathcal{T}}$, $\gamma_1' \in [\gamma']_{\mathcal{T}}$ such that $(\gamma_1, \gamma_1') \in \mathcal{T}^c$. Then γ_1 and γ_1' carry different orders, so the colours of $[\gamma]_{\mathcal{T}}$ and $[\gamma']_{\mathcal{T}}$ are different.

Corollary 7. *Let Φ be a translation invariant nearest neighbour interaction on $\Gamma = \mathbb{Z}^d$. If the Gibbs state for Φ is unique, then it has the global Markov property.*

Proof. Given Φ as above, define $i \in A$ if $(0, e_i) \in \mathcal{A}$; similarly define $i \in I$ and $i \in R$. Then clearly, for

$$\mathcal{T} = \mathcal{A} = \{(\gamma, \gamma + e_i) \mid \mathcal{T} \in \mathbb{Z}^d, i \in A\},$$

we get $\Gamma/\mathcal{T} \cong \mathbb{Z}^{d - \# A}$, which clearly is 2-colourable.

The $\leqslant$-maximal and $\leqslant$-minimal Gibbs states for the corresponding order (in the case of uniqueness: the unique Gibbs state) now have the global Markov property by fact 1.

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